Existence of solutions and other properties for an internal wave model

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The Nonlinear ILW System

We consider the nonlinear ILW system

$$\begin{aligned} \eta_t &= [(1-\alpha\eta)u]_x \\ u_t &= \eta_x - \alpha u u_x \end{aligned}$$

with initial data

$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x).$$

t denote the time variable and the spatial variable x stands in

$$\mathbb{R} \quad or \quad \mathbb{T} = [-\pi, \pi] \text{ (periodic functions).}$$

Questions

1. Existence of local solutions?
2. Blow up? or Global solutions?
3. Other properties?

A Friendly System

A system with more properties is given by

$$\begin{aligned} \eta_t &= [(1-\alpha\eta)u]_x \\ u_t &= \eta_x - \alpha u u_x + M(u) + N(u) \end{aligned}$$

where $M(u)$ denote dispersive terms and $N(u)$ denote dissipative terms.

- Some dispersive terms are

$$u_{xxt}, \quad H(u_{xt}), \quad \mathcal{T}(u_{xt}), \quad u_{xxx}, \quad \text{etc.}$$

- Some dissipative terms are

$$\delta u_{xx}, \quad H(u_x), \quad \mathcal{T}(u_x), \quad -u_{xxxx}, \quad \text{etc.}$$

Our problem

We will study the following nonlinear system

$$\begin{aligned} \eta_t &= [(1-\alpha\eta)u]_x \\ u_t &= \eta_x - \alpha u u_x + \delta u_{xx} \end{aligned}$$

with initial data

$$\eta(x,0)=\eta_0(x), \quad u(x,0)=u_0(x).$$

where the spatial variable stands at

$$x \in \mathbb{T} = [-\pi,\pi] \quad \text{(periodic solutions)}.$$

Questions

1. Existence of local solutions?
2. Blow up? or Global solutions?
3. Smoothing properties?
4. Asymptotic behavior?
5. Who is the limit?
6. Speed of this asymptotic behavior?

The nonlinear system can be written in the abstract framework

$$\frac{d}{dt}U = \mathbb{A}_0 U + \mathbb{A}\mathbb{F}(U),$$

where

$$U = \begin{pmatrix} \eta \\ u \end{pmatrix}, \quad \mathbb{A} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix}, \quad \mathbb{A}_0 = \begin{bmatrix} 0 & \partial_x \\ \partial_x & \delta\partial_{xx} \end{bmatrix},$$

$$\mathbb{F}(U) = \begin{pmatrix} -\alpha\eta u \\ -\alpha u^2/2 \end{pmatrix}.$$

We will work in the space

$$\mathbb{H}^s = H^s(\mathbb{T}) \times H^s(\mathbb{T}), \quad s \geq 0$$

with the the inner product

$$\langle U_1, U_2 \rangle = \langle \eta_1, \eta_2 \rangle_{H^s(\mathbb{T})} + \langle u_1, u_2 \rangle_{H^s(\mathbb{T})}.$$

This operators are defined in

$$D(\mathbb{A}) = \mathbb{H}^{s+1}, \quad D(\mathbb{A}_0) = \{U \in \mathbb{H}^s : \mathbb{A}_0(U) \in \mathbb{H}^s\}$$

and

$$D(\mathbb{F}) = \mathbb{H}^s \quad (\text{for } s > 1/2).$$

Note that $D(\mathbb{A}_0)$ is characterized by

$$D(\mathbb{A}_0) = \{U \in \mathbb{H}^s : u \in H^{s+1}(\mathbb{T}), \eta + \delta u_x \in H^{s+1}(\mathbb{T})\}.$$

is easy to see that

$$H^{s+1}(\mathbb{T}) \times H^{s+2}(\mathbb{T}) \subset D(\mathbb{A}_0) \subset \mathbb{H}^s.$$

Therefore A_0 is densely defined.

Some properties of the linearized system are preserved by the nonlinear system. In this case the linearized system is

$$\begin{aligned}\eta_t &= u_x,\\ u_t &= \eta_x + \delta u_{xx}.\end{aligned}$$

This system can be written in the abstract framework by

$$\frac{d}{dt}U = \mathbb{A}_0 U.$$

Theorem

$\mathbb{A}_0$ is the generator of a contractions semigroup.

We use the following result:

Theorem (Lumer-Phillips)

Let $\mathbb{A}$ be a operator in a Hilbert space $\mathbb{X}$. Then, $\mathbb{A}$ is the generator of a contractions semigroup if and only if it is densely defined and m-dissipative.

Definition: $\mathbb{A}$ is m-dissipative if

$$\text{Re}\langle \mathbb{A}U, U\rangle_{\mathbb{X}} \leq 0 \quad \text{and} \quad \text{Im}(\lambda I - \mathbb{A}) = \mathbb{X}$$

for some $\lambda > 0$.

Proof

We have

$$\operatorname{Re}\langle \mathbb{A}_0 U, U\rangle = -\delta \|u_x\|^2 < 0,$$

then, the operator $\mathbb{A}_0$ is dissipative. Let $F \in \mathbb{H}^s$, solving the equation $(I - A_0)U = F$ is equivalent to solve $\widehat{(I - A_0)U} = \widehat{F}$. If $\mu = \hat{\eta}$, $\omega = \hat{u}$ is equivalent to solve the system

$$\begin{aligned} \mu - ik\omega &= f \\ \omega - ik(\mu + i\delta k\omega) &= g \end{aligned}$$

where $(f, g) = \widehat{F}$. The solutions are

$$\mu = \frac{(1 + \delta k^2)f + ikg}{1 + \delta k^2 + k^2}, \quad \omega = \frac{g + ikf}{1 + \delta k^2 + k^2}.$$

after some computations we verify $U \in D(\mathbb{A}_0)$.

With the above theorem for each initial data $U_0 \in D(\mathbb{A}_0)$, we have a unique global solutions $U(t) = e^{t\mathbb{A}_0}U_0$ for the linearized system in the space

$$U \in C([0,\infty[, D(\mathbb{A}_0)) \cap C^1([0,\infty[, \mathbb{H}^s).$$

Theorem

The semigroup $\{e^{t\mathbb{A}_0}\}_{t\geq 0}$ *is analytic.*

we use the following result:

Theorem (a particular case of this theorem is proved in Liu's book)

Let $\mathbb{A}$ *the generator of a contractions semigroup* $\{e^{t\mathbb{A}}\}_{t\geq 0}$. *If the following conditions*

1. $\rho(\mathbb{A}) \supset i\mathbb{R}\setminus\{0\}$ *and*
2. $\|R(i\lambda,\mathbb{A})\| \leq \frac{C}{|\lambda|}$, *for all* $\lambda \in \mathbb{R}$, $\lambda \neq 0$

are satisfied, then $\{e^{t\mathbb{A}}\}_{t\geq 0}$ *is an analytic semigroup.*

Proof

We will use the discrete Fourier transform to show this theorem. If (μ,ω) denote the Fourier transform of $U=(\eta,u)$. The system $(i\lambda I-\mathbb{A}_0)U=F$, for $\lambda\in\mathbb{R}$ is satisfied if

$$\begin{aligned} i\lambda\mu - ik\omega &= f, \\ i\lambda\omega - ik(\mu + i\delta k\omega) &= g. \end{aligned}$$

Solving this equations we have

$$\mu = -\frac{i(\lambda - i\delta k^2)f + ikg}{\lambda^2 - i\delta k^2\lambda + k^2}, \quad \omega = -\frac{i\lambda g + ikf}{\lambda^2 - i\delta k^2\lambda + k^2}.$$

After some computations we have

$$\|U\| \le \frac{C}{|\lambda|}\|F\|, \quad \lambda \neq 0.$$

From this estimate we conclude that

$$\rho(\mathbb{A}_0) \supset i\mathbb{R}\setminus\{0\} \quad \text{and} \quad \|\lambda(i\lambda I - \mathbb{A}_0)^{-1}\| \le C$$

Returning to the nonlinear system

Applying the technique of parameters variations, the solution of the nonlinear system must satisfy the Duhamel's formula

$$U(t) = e^{t\mathbb{A}_0}U_0 + \int_0^t e^{(t-s)\mathbb{A}_0}\mathbb{A}\mathbb{F}(U(s))\, ds$$

If we consider the operator

$$(GU)(t) = e^{t\mathbb{A}_0}U_0 + \int_0^t e^{(t-s)\mathbb{A}_0}\mathbb{A}\mathbb{F}(U(s))\, ds$$

we use some theorem of fixed point to find solutions of the nonlinear system.

Because $\mathbb{A}\mathbb{A}_0 = \mathbb{A}_0\mathbb{A}$ in $D(\mathbb{A}_0) \cap D(\mathbb{A})$ we can verify that $\mathbb{A}$ commute with $e^{(t-s)\mathbb{A}_0}$ em $D(\mathbb{A})$, and in this case

$$
\begin{aligned}
(GU)(t) &= e^{t\mathbb{A}_0}U_0 + \int_0^t \mathbb{A}e^{(t-s)\mathbb{A}_0}\mathbb{F}(U(s))\,ds \\
&= e^{s\mathbb{A}_0}U_0 + \int_0^t \mathbb{A}e^{s\mathbb{A}_0}\mathbb{F}(U(t-s))\,ds.
\end{aligned}
$$

Difficulty: unfortunately the operator $\mathbb{A}e^{t\mathbb{A}_0}$ blow up at $t = 0$. I explain: it can be show that the function $t \to \mathbb{A}e^{t\mathbb{A}_0}$ are continuous in $L(\mathbb{H}^s)$ for $t > 0$. The blow up is consequence of this fact

$$
\mathbb{A}e^{t\mathbb{A}_0} \to \mathbb{A}, \quad \text{when} \quad t \to 0^+ \quad \text{and} \quad \mathbb{A} \notin L(\mathbb{H}^s).
$$

Therefore we need some kind control on $\|\mathbb{A}e^{t\mathbb{A}_0}\|$ near to $t = 0$.
It is known that $\mathbb{A}_0 e^{t\mathbb{A}_0}$ is a limited operators for $t > 0$ and

$$\|\mathbb{A}_0 e^{t\mathbb{A}_0}\| \leq \frac{C}{t}, \quad t > 0,$$

Because the operator $\mathbb{A}$ is more “weak” than $\mathbb{A}_0$, it is possible to show the same inequality, that is

$$\|\mathbb{A}e^{t\mathbb{A}_0}\| \leq \frac{C}{t}, \quad t > 0,$$

but this do not help me.

Theorem

There exist $\theta \in]0,1[$ *such that*

$$\|\mathbb{A}e^{t\mathbb{A}_0}\| \leq \frac{C}{t^\theta}$$

Corollary

we have the following estimate

$$\int_0^t \|\mathbb{A}e^{s\mathbb{A}_0}\| \, ds \leq \frac{C}{1-\theta} t^{1-\theta}, \quad \forall t > 0.$$

Some estimates for the nonlinear term

$$\mathbb{F}(U) = \begin{pmatrix} -\alpha\eta u \\ -\alpha u^2/2 \end{pmatrix} \quad \Rightarrow \quad D\mathbb{F}(U) = \begin{bmatrix} -\alpha u & -\alpha\eta \\ 0 & -\alpha u \end{bmatrix}$$

Therefore $\|D\mathbb{F}(U)\| \leq C\|U\|$. Since

$$\mathbb{F}(U_2) - \mathbb{F}(U_1) = \int_0^1 D\mathbb{F}\big(U_1 + r(U_2 - U_1)\big)(U_2 - U_1)\, dr,$$

it follows that

$$\|\mathbb{F}(U_2) - \mathbb{F}(U_1)\| \leq C(\|U_1\| + \|U_2\|)\|U_2 - U_1\|.$$

Consequently, if $\|U_1 - U_0\| \leq R$, $\|U_2 - U_0\| \leq R$ for some U_0, we have

$$\|\mathbb{F}(U_2) - F(U_1)\| \leq C(R + \|U_0\|)\|U_2 - U_1\|.$$

Theorem (Local solutions)

For $U_0 \in D(\mathbb{A}_0)$, the nonlinear system has a unique solution em $C([0,T],\mathbb{H}^s)$ for some $T > 0$

Proof

Let $T > 0$, $R > 0$, we consider the subset of the space $C([0,T],\mathbb{H}^s)$:

$$\mathbb{M}_T = \left\{ U \in C([0,T],\mathbb{H}^s) : U(0) = U_0,\ U(t) \in \overline{B_R(U_0)} \right\}.$$

We define the operator $G : \mathbb{M}_T \to C([0,T],\mathbb{H}^s)$ given by

$$G(U)(t) \quad = \quad e^{t\mathbb{A}_0} U_0 + \int_0^t \mathbb{A} e^{s\mathbb{A}_0} F(U(t-s))\, ds.$$

we have

$$\begin{aligned}
&\|G(U)(t) - U_0\| \\
&\quad\leq \quad \|e^{t\mathbb{A}_0}U_0 - U_0\| + \int_0^t \|\mathbb{A}e^{s\mathbb{A}_0}\| \left(\|F(U(t-s)) - F(U_0)\| + \|F(U_0\|\right) \, ds \\
&\quad\leq \quad \|e^{t\mathbb{A}_0}U_0 - U_0\| + \left(\int_0^t \|\mathbb{A}e^{s\mathbb{A}_0}\| \, ds\right) \{CR(R + \|U_0\|) + \|F(U_0\|\} \\
&\quad\leq \quad \|e^{t\mathbb{A}_0}U_0 - U_0\| + C(\theta)t^{1-\theta} \{CR(R + \|U_0\|) + \|F(U_0\|\} .
\end{aligned}$$

Taking T small we have

$$\|G(U)(t) - U_0\| \leq R, \; \forall t \in [0, T].$$

This shows that $G(\mathbb{M}_T) \subset \mathbb{M}_T$ for T small.

On the other hand,

$$\begin{aligned}\|G(U_2)(t) - G(U_1)(t)\| \\ &\leq \int_0^t \|\mathbb{A}e^{s\mathbb{A}_0}\| \|F(U_2(t-s)) - F(U_1(t-s))\| \, ds \\ &\leq \left(\int_0^t \|\mathbb{A}e^{s\mathbb{A}_0}\| \, ds\right) C(R + \|U_0\|) \|U_2 - U_1\|_{C([0,T],\mathbb{H}^s)} \\ &\leq C(\theta) t^{1-\theta} (R + \|U_0\|) \|U_2 - U_1\|_{C([0,T],\mathbb{H}^s)}.\end{aligned}$$

Taking T small we have that G is a contraction operator.

Questions

1. Existence of local solutions? OK
2. Blow up? or Global solutions? still trying!
3. Smoothing properties? it's possible
4. Asymptotic behavior?
5. Who is the limit?
6. Speed of this asymptotic behavior?

Asymptotic behavior

If $(\eta_\infty(x), u_\infty(x))$ is the limit of the solutions $(\eta(x,t), u(x,t))$ when $t \to \infty$, then (η_∞, u_∞) is the solution of the stationary system

$$\begin{aligned} [(1-\alpha\eta)u]_x &= 0, \\ \eta_x - \alpha u u_x + \delta u_{xx} &= 0. \end{aligned}$$

But the solutions of this this system are constants. Therefore η_∞, u_∞ are constants. On the other hand, from the preserved amounts

$$\int_{-\pi}^{\pi} \eta(x,t)\,dx = \int_{-\pi}^{\pi} \eta_0(x)\,dx, \quad \int_{-\pi}^{\pi} u(x,t)\,dx = \int_{-\pi}^{\pi} u_0(x)\,dx$$

we conclude that

$$\eta_\infty = \frac{1}{2\pi}\int_{-\pi}^{\pi} \eta_0(x)\,dx, \quad u_\infty = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(x)\,dx.$$

If we introduce the notation

$$\tilde{h} = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x)\, dx,$$

the space $H_0^s(\mathbb{T}) = \{h \in H^s(\mathbb{T}) : \tilde{h} = 0\}$ is a closed subspace of $H^s(\mathbb{T})$. Consequently, the space

$$\mathbb{H}_0^s = H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$$

is a closed subspace of $\mathbb{H}^s$. Therefore, it is Hilbert subspace.

Now, note that, if $(\eta, u) \in \mathbb{H}^s$ is a solution of the nonlinear system if and only if $(\eta - \tilde{\eta}_0, u - \tilde{u}_0)$ is a solution of the following auxiliary nonlinear system

$$\begin{aligned} \eta_t &= \beta_1 \eta_x + \beta_2 u_x - \alpha(\eta u)_x, \\ u_t &= \beta_3 \eta_x + \beta_4 u_x - \alpha u u_x + \delta u_{xx}, \end{aligned}$$

where

$$\beta_1 = \beta_4 = -\alpha \tilde{u}_0, \quad \beta_2 = 1 - \alpha \tilde{\eta}_0, \quad \beta_3 = 1.$$

Moreover, we have

$$(\eta - \tilde{\eta}_0, u - \tilde{u}_0) \in \mathbb{H}^s_0.$$

The auxiliary system can be write as

$$\frac{d}{dt}U = \mathbb{A}_0 U + \mathbb{A}\mathbb{F}(U)$$

where

$$\mathbb{A} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix}, \quad \mathbb{A}_0 = \begin{bmatrix} \beta_1\partial_x & \beta_2\partial_x \\ \beta_3\partial_x & \beta_4\partial_x + \delta\partial_{xx} \end{bmatrix}, \quad \mathbb{F}U = \begin{pmatrix} -\alpha\eta u \\ -\alpha u^2/2, \end{pmatrix}$$

with domains

$$D(\mathbb{A}) = \mathbb{H}^{s+1}, \quad D(\mathbb{A}_0) = \{U \in \mathbb{H}^s : \mathbb{A}_0(U) \in \mathbb{H}^s\}$$
$$D(\mathbb{F}) = \mathbb{H}^s \quad (s > 1/2)$$

Properties of the new operator $\mathbb{A}_0$

1. $\mathbb{A}_0$ is a semigroup of contractions $\{e^{t\mathbb{A}_0}\}_{t\geq 0}$ in the space $\mathbb{H}^s$ with the appropriate inner product.
2. The subspace $\mathbb{H}_0^s$ is invariant for this semigroup. That is, $e^{t\mathbb{A}_0}(\mathbb{H}_0^s) \subset \mathbb{H}_0^s$.
3. The semigroup $\{e^{t\mathbb{A}_0}\}_{t\geq 0}$ is analytic in $\mathbb{H}^s$.
4. The semigroup $\{e^{t\mathbb{A}_0}\}_{t\geq 0}$ is analytic and exponentially stable in $\mathbb{H}_0^s$. ($\|e^{t\mathbb{A}_0}\| \leq Me^{-\gamma t}$, $\gamma > 0$).
5. $\|\mathbb{A}e^{t\mathbb{A}_0}\| \leq \dfrac{C}{t^\theta}$, for some $\theta \in]0,1[$

Thus, we have the local solutions for the auxiliary system in $\mathbb{H}^s$.

Global solution? still trying!

Theorem

For the solutions of the nonlinear system we have

$$\|\eta-\tilde{\eta}_0\|_{H^s(\mathbb{T})}+\|u-\tilde{u}_0\|_{H^s(\mathbb{T})}\leq Ce^{-\gamma t}.$$

where the constant C depends of the initial data.

Proof

We multiply the auxiliary system by $e^{\gamma t}$. Thus, the functions $(e^{\gamma t}\eta, e^{\gamma t}u)$ satisfy the following system

$$\begin{aligned} \eta_t &= (\beta_1+\gamma)\eta_x+\beta_2 u_x-\alpha e^{-\gamma t}(\eta u)_x, \\ u_t &= \beta_3\eta_x+(\beta_4+\gamma)u_x-\alpha e^{-\gamma t}uu_x+\delta u_{xx}. \end{aligned}$$

The operator of the linear part of this system is

$$\mathbb{A}_\gamma=\begin{bmatrix}(\beta_1+\gamma)\partial_x & \beta_2\partial_x \\ \beta_3\partial_x & (\beta_4+\gamma)\partial_x+\delta\partial_{xx}\end{bmatrix}.$$

The above system can be written in the abstract framework

$$\frac{d}{dt}U = \mathbb{A}_\gamma U + e^{-\gamma t}\mathbb{A}F(U)$$

$\mathbb{A}_\gamma$ has the same properties fo $\mathbb{A}_0$ for γ small. From Duhamel's formula we have

$$U(t) = e^{t\mathbb{A}_\gamma}U_0 + \int_0^t e^{-\gamma s}e^{(t-s)\mathbb{A}_\gamma}\mathbb{A}F(U(s))\,ds.$$

If for the initial data

$$U_0 = \begin{pmatrix} \eta_0 - \tilde{\eta}_0 \\ u_0 - \tilde{u}_0 \end{pmatrix}$$

We will show that this system has global solution and the solution is bounded, that is, $\|U(t)\| \leq C$, for all $t \geq 0$. This means that

$$\|e^{\gamma t}(\eta - \tilde{\eta}_0)\|_{H^s(\mathbb{T})} + \|e^{\gamma t}(u - \tilde{u}_0)\|_{H^s(\mathbb{T})} \leq C.$$

References

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