An introduction to Gevrey Spaces.

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Seminars on PDE's and Analysis

The Heat Operator

Let L be the heat operator

$$L = \frac{\partial}{\partial_x} - \frac{\partial^2}{\partial_{t^2}},\ (t,x) \in \mathbb{R}^2.$$

Its fundamental solution ($L(E) = \delta$) is given by

$$E(t,x) = \begin{cases} (4\pi x)^{-1/2} e^{-t^2/4x}, & \text{if } x > 0, \\ 0, & \text{if } x \leqslant 0, \end{cases}$$

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- Then, there are solutions of the homogeneous equation $Lu=0$ which are not analytic in general, though always C^∞;
- Fixed a compact $K\subset\mathbb{R}^2$, $0\notin K$, we obtain

$$|\partial^\alpha E(x)| \leqslant C^{|\alpha|+1}(\alpha!)^2.$$

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3. The Gevrey spaces $G^s(\mathbb{T}^n)$:
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4. Some applications to Global Regularity in G^s;

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4. $C^\omega(\mathbb{T}^n)$ is the space of functions in $C^\infty(\mathbb{T}^n)$ such that $\exists c > 0$ with

$$\sup_{x\in\mathbb{T}^n} |\partial^\alpha f(x)| \leqslant C^{|\alpha|+1}\alpha!$$

Fourier Series on $C^\infty(\mathbb{T}^n)$

Given $f \in C^\infty(\mathbb{T}^n)$ we can write

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) e^{ix\cdot\xi},$$

where $\widehat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-ix\cdot\xi} f(x) dx.$

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Theorem

Given a sequence $\{c_\xi\}_{\xi \in \mathbb{Z}^n}$ we obtain an element

$$f = \sum_{\xi \in \mathbb{Z}^n} c_\xi e^{ix\cdot\xi} \in C^\infty(\mathbb{T}^n)$$

if, and only if, for each $N \in \mathbb{N}$ there exists $C > 0$ such that

$$|c_\xi| \leqslant C|\xi|^{-N}, \ |\xi| \to \infty.$$

By $\mathcal{D}'(\mathbb{T}^n)$ we set the space of continuous linear operators $u : \mathbb{C}^\infty(\mathbb{T}^n) \to \mathbb{C}$, where continuous mean: $\exists C > 0$ and $m \in \mathbb{N}$ such that

$$|u(\varphi)| \leqslant C \sup_{\alpha \leqslant m} \sup_{x \in \mathbb{T}^n} |\partial^\alpha \varphi(x)|, \ \forall \varphi \in C^\infty(\mathbb{T}^n).$$

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Example

If $f \in L^p(\mathbb{T}^n)$ we obtain $u_f \in \mathcal{D}'(\mathbb{T}^n)$ by defining

$$u_f(\varphi) = \int_{\mathbb{T}^n} f(x)\varphi(x)dx, \ \varphi \in C^\infty(\mathbb{T}^n).$$

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Example

If $u \in \mathcal{D}'(\mathbb{T}^n)$ we obtain $\partial^\alpha u \in \mathcal{D}'(\mathbb{T}^n)$ by defining

$$\partial^\alpha u = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$$

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Given a sequence $\{c_\xi\}_{\xi\in\mathbb{Z}^n}$ we obtain an element

$$u = \sum_{\xi \in \mathbb{Z}^n} c_\xi e^{ix\cdot\xi} \in \mathcal{D}'(\mathbb{T}^n)$$

if, and only if, there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$|c_\xi| \leqslant C|\xi|^N, \ |\xi| \to \infty.$$

Hypoellipticity

Let m be a positive integer and P the partial differential operator

$$P = P(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial_x^\alpha, \; x \in \mathbb{T}^n,$$

where α is a multindex in $\mathbb{Z}_+^n$. We say that $P(x, D)$ is:

(GH) if conditions

$$u \in \mathcal{D}'(\mathbb{T}^n) \;\text{ and }\; Pu \in C^\infty(\mathbb{T}^n) \;\text{ imply }\; u \in C^\infty(\mathbb{T}^n);$$

(GAH) if conditions

$$u \in \mathcal{D}'(\mathbb{T}^n) \;\text{ and }\; Pu \in C^\omega(\mathbb{T}^n) \;\text{ imply }\; u \in C^\omega(\mathbb{T}^n);$$

Constant coefficient case

Let $P(D) = \sum_{|\alpha|\leqslant m} a_\alpha \partial_x^\alpha,\ a_\alpha$ be a constant coefficient operator and

$$P(\xi) = \sum_{|\alpha|\leqslant m} a_\alpha \xi^\alpha$$

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Theorem (Greenfield and Wallach (1972))

The operator $P(D)$ is (GH) if, and only if, there are $C, M, R > 0$ such that

$$|P(\eta)| \geqslant C|\eta|^M,\ |\eta| \geqslant R.$$

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Theorem (Greenfield (1972))

The operator $P(D)$ is (GAH) if, and only if, for each $\epsilon>0$ there exists $C_\epsilon>0$

$$|P(\eta)|\geqslant e^{-\epsilon|\eta|}, |\eta|\geqslant C_\epsilon.$$

Approximation by Rational Numbers

Theorem

The differential operator

$$P = D_t + (\alpha + i\beta)D_x,\ \omega \in \mathbb{C},\ (t,x) \in \mathbb{T}^2.$$

is (GH) iff, either

(i) $\beta \neq 0$, *or*

(ii) α *is an irrational non-Liouville number, that is,* $\exists C, M > 0$ *such that*

$$|m + \alpha n| \geqslant C|(m,n)|^M,\ \forall (m,n) \in \mathbb{Z}^2.$$

Exponential Approximation by Rational Numbers

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3. There exist $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that

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is neither (GH) nor (GAH).

Gevrey class

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For $s \in \mathbb{R}$, with $s \geqslant 1$, we say that a smooth function $f \in C^\infty(\mathbb{T}^n)$ is in the Gevrey class $G^s(\mathbb{T}^n)$ if there exists $C > 0$ such that

$$\sup_{x \in \mathbb{T}^n} |\partial^\alpha f(x)| \leqslant C^{|\alpha|+1}(\alpha!)^s, \ \forall \alpha \in \mathbb{Z}_+^n.$$

In particular, $G^s(\mathbb{T}^n)$ is a Banack space endowed with the norm

$$\|f\|_s = \sup_{\alpha \in \mathbb{Z}_+^n} \left\{ \sup_{x \in \mathbb{T}^n} |\partial^\alpha f(x)| (\alpha!)^{-s} \right\};$$

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- For $s = 1$ we have $G^1(\mathbb{T}^n) = C^\omega(\mathbb{T}^n)$. Moreover, since

$$1 \leqslant s < t \ \text{ imply } \ G^s(\mathbb{T}^n) \subsetneq G^t(\mathbb{T}^n),$$

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we obtain $C^\omega(\mathbb{T}^n) \subset G^s(\mathbb{T}^n)$, fora all $s \geqslant 1$;
- If $f \in G^s(\mathbb{T}^n)$, then $\partial^\alpha f \in G^s(\mathbb{T}^n)$, for any $\alpha \in \mathbb{Z}_+^n$;
- $G^1(\mathbb{T}^n) \subsetneq \bigcap_{s>1} G^s(\mathbb{T}^n)$;
- $\bigcup_{s>1} G^s(\mathbb{T}^n) \subsetneq C^\infty(\mathbb{T}^n)$;

The space $\mathcal{D}'_s(\mathbb{T}^n)$

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By $\mathcal{D}'_s(\mathbb{T}^n)$ we set the space of continuous linear operators $u : G^s(\mathbb{T}^n) \to \mathbb{C}$, called the space of ultradistributions.

Example

Given $m \in \mathbb{Z}^+$ consider $u = \sum_{|\alpha| \leqslant m} a_\alpha \delta^\alpha$, with $a_\alpha \in \mathbb{C}$, defined formally by defining

$$u(\varphi) = \sum_{|\alpha| \leqslant m} a_\alpha \partial^\alpha \varphi(0), \ \varphi \in G^s(\mathbb{T}^n).$$

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- Since $G^s(\mathbb{T}^n)$ is dense in C^∞, $s \geqslant 1$, it follows that

$$\mathcal{D}'(\mathbb{T}^n) \subsetneq \mathcal{D}'_s(\mathbb{T}^n).$$

The proper inclusion can be obtained by the last example, since $u \notin \mathcal{S}'(\mathbb{T}^n) \subset \mathcal{D}'(\mathbb{T}^n)$. In particular, justifying the name **ultradistribution**.

Theorem

Let $\{C_\xi\}_{\xi\in\mathbb{Z}^n} \subset \mathbb{C}$*, satisfying*

$$|C_\xi| \leqslant Ce^{-\varepsilon|\xi|^{1/s}}, \forall\xi \in \mathbb{Z}^n,$$

for some $\varepsilon > 0$ *and* $C > 0$*. Then, there exists* $f \in G^s(\mathbb{T}^n)$ *with*

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} C_\xi e^{ix\cdot\xi}.$$

Moreover, $\widehat{f}(\xi) = C_\xi$*.*

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Theorem

If either $u \in \mathcal{D}_s'(\mathbb{T}^n)$*, or* $u \in \mathcal{D}'(\mathbb{T}^n)$*, then* $u \in G^s(\mathbb{T}^n)$ *if, and only if, there exists* $\varepsilon > 0$ *and* $c_\varepsilon > 0$

$$|c_\xi| \leqslant c_\varepsilon e^{-\varepsilon|\xi|^{1/s}}, \ \forall \xi \in \mathbb{Z}^n$$

Hypoellipticity

A partial differential operator $P(x,D)=\sum_{|\alpha|\leqslant m} a_\alpha(x)\partial_x^\alpha$, with $a_\alpha \in C^\omega(\mathbb{T}^n)$, is said to be globally G^s hypoelliptic (G^sH) if

$$u \in \mathcal{D}_s'(\mathbb{T}^n) \text{ and } Pu \in G^s(\mathbb{T}^n) \text{ imply } u \in G^s(\mathbb{T}^n);$$

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Remark

We can replace this definition by

$$u \in \mathcal{D}'(\mathbb{T}^n) \textit{ and } Pu \in G^s(\mathbb{T}^n) \textit{ imply } u \in G^s(\mathbb{T}^n),$$

which is more weaker.

Constant coefficient case

Theorem

The operator $P(D) = \sum_{|\alpha| \leqslant m} a_\alpha \partial_x^\alpha$, a_α *is* $G^s H$ *if, and only if, for each* $\epsilon > 0$ *there exists* $C_\epsilon > 0$

$$|P(\xi)| \geqslant e^{-\epsilon |\xi|^{1/s}}, |\xi| \geqslant C_\epsilon.$$

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Theorem

The operator $P(D) = \sum_{|\alpha|\leqslant m} a_\alpha \partial_x^\alpha,\ a_\alpha$ *is* G^sH *if, and only if, for each* $\epsilon > 0$ *there exists* $C_\epsilon > 0$

$$|P(\xi)| \geqslant e^{-\epsilon|\xi|^{1/s}}, |\xi| \geqslant C_\epsilon.$$

Example

The operator $P = D_t - \alpha D_x$, $\alpha \in \mathbb{R}$, is G^sH if, and only if, $\epsilon > 0$ there exists $C_\epsilon > 0$

$$|m - \alpha n| \geqslant e^{-\epsilon|\xi|^{1/s}}, |\xi| \geqslant C_\epsilon.$$

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We observe that, if $1 \leqslant s < t$, then G^sH implies G^tH. Thus, we obtain

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- For the heat operator (introduction) he obtain solutions of $Lu = 0$ that are not C^ω, but are in G^2;
- There exist equations $P(D)u = f \in G^s$ without solutions in $\mathcal{D}'$ with solutions in $\mathcal{D}'_s$;

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We point out that that $G^s H \not\Rightarrow C^\infty H$. The following theorem is very interesting:

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Theorem (Gramchev-Popivanov-Yoshino)

Fixed a number $1 \leqslant \sigma < \infty$, there exist a real-valued function $a \in C^\omega$ such that

$$P = D_t - a(t)D_x$$

is G^sH, for $1 \leqslant s \leqslant \sigma$, while is not G^sH for $1 \leqslant \sigma < s$.

- Himonas and Petronilho have works in the investigation of the problem

$$GH \ \Rightarrow \ G^s H,$$

for operators with non-constant coefficients. For instance, they solve the problem for

$$P = \partial_t^2 + [\partial_x - a(t)\partial_y]^2, \ (t,x,y) \in \mathbb{T}^3.$$

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Thank you !!!